The Embedding Structure for LOTS

Alex Primavesi

University of East Anglia

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Joint work with Katie Thompson.

What is a LOTS?

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LOTS embeddings

A LOTS embedding is a linear order embedding that is also continuous. In this case not only do we get $B' \cong A$ as before but also $\{f[u] : u \in \tau_A\} = \{B' \cap v : v \in \tau_B\}$, so it makes sense to say, again informally, that the LOTS B contains a copy of A.

We can quasi-order the class of all LOs/LOTS by setting $A \leq B$ if and only if A embeds/LOTS-embeds into B. Similarly, we can quasi-order the set of all LOs/LOTS of a given cardinality, κ. What are the consistent properties of these quasi-orders? How do they differ for LOTS and linear orders?

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Universal Linear Orders

Under GCH, the embedding quasi-order for linear orders of size κ has a unique (up to isomorphism) maximal element. For the countable case, this is the rationals $\mathbb{Q}$. For $\kappa \geq \omega_1$, this is a linear order generalising the density property of the rationals to a property called κ-*saturation*:

$$\forall S, T \in [\mathbb{Q}(\kappa)]^{<\kappa}\ [S < T \Rightarrow (\exists x) S < x < T].$$

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If we take the completion of $\mathbb{Q}(\kappa)$ under sequences of length less than κ, then to some extent we get around counterexamples like this. But again, $\omega + 1 + \omega^*$ cannot be continuously mapped into it. (We will denote this partial completion by $\bar{\mathbb{Q}}(\kappa)$, but note that there are still sequences of length κ with no sup/inf – so in particular it still has size κ.)

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(GCH) $\bar{\mathbb{Q}}(\kappa)$ has size κ and is universal for κ-entwined LOTS.

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Recall that a *linear continuum* is a linear order that is both *dense* and *complete*. The IVT tells us that if A is a linear continuum and $f : A \to B$ is continuous and order-preserving, then $f[A]$ must be a convex subset of B.

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Let $[0, 1] \subseteq \mathbb{R}$ *denote the closed unit interval – that is, a copy of* $\mathbb{R}$ *with endpoints – and* $[0, 1)$ *an isomorphic copy of* $\mathbb{R}$ *with a least point but no greatest point. Then each of the following is a linear continuum:*

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R_0 cannot be continuously embedded into R_1. Likewise, R_1 cannot be continuously embedded into R_0.

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By the I.V.T., if there was such an embedding then R_1 would contain an interval isomorphic to R_0, or vice versa. This is clearly not the case. □

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If $X, Y \in [\kappa]^\kappa$ are such that there is no $\alpha < \kappa$ with $X \setminus \alpha = Y \setminus \alpha$ then there is no LOTS embedding $f : R_X \to R_Y$. Thus we can find 2^κ many LOTS that are pairwise non-embeddable.

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The basis question

We have seen that the *top* of the LOTS embedding quasi-order is maximally complex for a final section of the class of cardinals, contrasting with the case for linear orders.

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A basis for uncountable LOTS

However, the five element basis is not a basis for the uncountable LOTS. But by adding a few carefully chosen linear orders we can get an *eleven* element basis for the uncountable LOTS under PFA. We can prove that this is the smallest possible basis that can exist for the uncountable LOTS in any model of ZFC.

The basis consists of:

A basis for uncountable LOTS

However, the five element basis is not a basis for the uncountable LOTS. But by adding a few carefully chosen linear orders we can get an *eleven* element basis for the uncountable LOTS under PFA. We can prove that this is the smallest possible basis that can exist for the uncountable LOTS in any model of ZFC.

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